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From (1) and (2),  $\cos 2\alpha = \frac{(c-\alpha)n}{e^2}$ .

From (3),  $\sin 2\alpha = \frac{-2bn}{e^2}$ .

These equations determine  $\alpha$  without ambiguity. Substituting for  $\alpha$ ,  $e$ ,  $n$  in the equations

$$h - e^2 p \cos \alpha = -dn,$$

$$k - e^2 p \sin \alpha = -en,$$

$$h^2 + k^2 - e^2 p^2 = fn,$$

and solving for  $h$ ,  $k$ ,  $p$ , the curve is completely determined.

The solution of these last equations will be much simpler if the given equation is first transformed to the center, for we will then have

$$h = e^2 p \cos \alpha, \quad k = e^2 p \sin \alpha, \quad h^2 + k^2 - e^2 p^2 = f'n,$$

$f'$  being obtained by substituting the coördinates of the center in the left hand member of the given equation.

If  $a'$  = semi-major axis,  $b'$  = semi-minor axis, we have  $a' = ep$ ,  $b' = a' \sqrt{1 - e^2}$ .

Let us take the equation  $3x^2 + 2xy + 3y^2 - 16y + 20 = 0$ .

Transformed to centre  $(-1, 3)$ ,  $3x^2 + 2xy + 3y^2 - 4 = 0$ , we find

$$e^2 = \frac{1}{2}, \quad n = \frac{2 - \frac{1}{2}}{6} = \frac{1}{4}, \quad \sin 2\alpha = -1, \quad \cos 2\alpha = 0.$$

$$\alpha = 135^\circ, \quad h = -\frac{p}{1'8}, \quad k = \frac{p}{1'8}, \quad p = \pm 2, \quad a' = 1'2, \quad b' = 1.$$

In general the equations for  $h$ ,  $k$ ,  $p$  give two values of each quantity showing that the conic has two directrices and two foci.

If  $e = 1$ , then  $h - p \cos \alpha = -dn$ ,  $k - p \sin \alpha = -en$ ,  $h^2 + k^2 - p^2 = fn$ . The terms containing  $p^2$  cancel, showing that the parabola has one directrix and one focus at infinity.

## INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Continued from April Number.]

$$\therefore B_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{d\varphi}{(1 + e^2 - 2e \cos \varphi)^{\frac{3}{2}}}$$

$$= \frac{4}{\pi(1-e^2)^2} [2E(e, \tfrac{1}{2}\pi) - (1-e^2)F(e, \tfrac{1}{2}\pi)] \dots\dots (64).$$

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos \varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}} \\ = \frac{4}{\pi e(1-e^2)^{\frac{3}{2}}} [(1+4e^2)E(e, \tfrac{1}{2}\pi) - (1+2e^2)F(e, \tfrac{1}{2}\pi)] \dots\dots (65).$$

$$C_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}} \\ = \frac{4}{3\pi(1-e^2)^{\frac{3}{2}}} [8(1+e^2)E(e, \tfrac{1}{2}\pi) - (5-2e^2-3e^4)F(e, \tfrac{1}{2}\pi)] \dots\dots (66).$$

$$C_1 = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos \varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}} \\ = \frac{4}{3\pi e(1-e^2)^{\frac{3}{2}}} [(1+14e^2-e^4)E(e, \tfrac{1}{2}\pi) - (1+6e^2-7e^4)F(e, \tfrac{1}{2}\pi)] \dots\dots (67).$$

$$D_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}} = \frac{4}{15\pi(1-e^2)^6} [2(23+82e^2 \\ +23e^4)E(e, \tfrac{1}{2}\pi) - (31+51e^2-67e^4-15e^6)F(e, \tfrac{1}{2}\pi)] \dots\dots (68).$$

These expressions are also useful in finding the *disturbing* force between two planets.

Before dealing with general expressions we will find expressions containing the Elliptic Integral of the third order.

$$\int_0^{2\pi} \frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} = \Pi(e, c, \tfrac{1}{2}\pi) \dots\dots\dots (69).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{c} \int_0^{\frac{1}{2}\pi} \frac{[(1+c\sin^2\theta)-1]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} \\ = \frac{1}{c} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{c} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} \\ = \frac{1}{c} [F(e, \tfrac{1}{2}\pi) - \Pi(e, c, \tfrac{1}{2}\pi)] \dots\dots\dots (70).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{(1-\sin^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{1}{c} [(c+1)II(e, c, \frac{1}{2}\pi) - F(e, \frac{1}{2}\pi)] \dots \dots \dots (71).$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{\sin^4 \theta d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} = \frac{1}{c^2} \int_0^{\frac{1}{2}\pi} \frac{[(1+\sin^2 \theta)^2 - 2(1+\sin^2 \theta + 1)] d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{c^2} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1+\sin^2 \theta)_1 (1-e^2 \sin^2 \theta)} + \frac{1}{c^2} \int_0^{\frac{1}{2}\pi} \frac{(\sin^2 \theta - 1) d\theta}{\sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{c^2 e^2} [(c-e^2)F(e, \frac{1}{2}\pi) - cE(e, \frac{1}{2}\pi) + e^2 II(e, c, \frac{1}{2}\pi)] \dots \dots \dots (72). \end{aligned}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{\cos^4 \theta d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} = \int_0^{\frac{1}{2}\pi} \frac{(1-2\sin^2 \theta + \sin^4 \theta) d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{c^2 e^2} [(c+1)^2 e^2 II(e, c, \frac{1}{2}\pi) - cE(e, \frac{1}{2}\pi) + (c-e^2-2ce^2)F(e, \frac{1}{2}\pi)] \dots (73). \end{aligned}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{\sin^2 \theta \cos^2 \theta d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} = \int_0^{\frac{1}{2}\pi} \frac{(\sin^2 \theta - \sin^4 \theta) d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{c^2 e^2} [cE(e, \frac{1}{2}\pi) + (ce^2 + e^2 - c)F(e, \frac{1}{2}\pi) - (c+1)e^2 II(e, c, \frac{1}{2}\pi)] \dots \dots \dots (74). \end{aligned}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{\sin^6 \theta d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} = \frac{1}{c^3} \int_0^{\frac{1}{2}\pi} \frac{[(1+\sin^2 \theta)^3 - (1+3\sin^2 \theta + \sin^4 \theta)] d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{3 c^3 e^4} [(3e^4 - 3ce^2 + c^2 e^2 + 2c^2)F(e, \frac{1}{2}\pi) - 3e^4 II(e, c, \frac{1}{2}\pi) \\ & \quad - (2c^2 + 2c^2 e^2 - 3ce^2)E(e, \frac{1}{2}\pi)] \dots \dots \dots (75). \end{aligned}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{\sin^4 \theta \cos^2 \theta d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} = \int_0^{\frac{1}{2}\pi} \frac{(\sin^4 \theta - \sin^6 \theta) d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{3 c^3 e^4} [3e^4 (c+1)II(e, c, \frac{1}{2}\pi) + (2c^2 - c^2 e^2 - 3ce^2)E(e, \frac{1}{2}\pi) \\ & \quad - (3e^4 - 3ce^2 + 2c^2 - 2c^2 e^2 + 3ce^4)F(e, \frac{1}{2}\pi)] \dots \dots \dots (76). \end{aligned}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}\pi} \frac{\sin^2 \theta \cos^4 \theta d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} + \int_0^{\frac{1}{2}\pi} \frac{(\sin^2 \theta \cos^2 \theta - \sin^4 \theta \cos^2 \theta) d\theta}{(1+\sin^2 \theta)_1 \sqrt{(1-e^2 \sin^2 \theta)}} \\ &= \frac{1}{3 c^3 e^4} [(3e^4 + 3c^2 e^4 - 3ce^2 + 6ce^4 - 5c^2 e^2 + 2c^2)F(e, \frac{1}{2}\pi) \\ & \quad - 3e^4 (c+1)^2 II(e, c, \frac{1}{2}\pi) - (2c^2 - 4c^2 e^2 - 3ce^2)E(e, \frac{1}{2}\pi)] \dots \dots \dots (77). \end{aligned}$$

$$\begin{aligned}
& \int_0^{\frac{1}{2}\pi} \frac{\cos^6 \theta d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} = \int_0^{\frac{1}{2}\pi} \frac{(\cos^4 \theta - \sin^2 \theta \cos^4 \theta) d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} \\
& = \frac{1}{3c^3 e^4} [3e^4(c+1)^3 II(e, c, \tfrac{1}{2}\pi) + (2c^3 - 7c^2 e^2 - 3ce^2) E(e, \tfrac{1}{2}\pi) \\
& \quad - (3e^4 + 9c^2 e^4 - 3ce^2 + 9ce^4 - 8c^2 e^2 + 2c^2) F(e, \tfrac{1}{2}\pi)] \dots \dots \dots (78).
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\frac{1}{2}\pi} \frac{\sin^8 \theta d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} \\
& = \frac{1}{c^4} \int_0^{\frac{1}{2}\pi} \frac{[(1+\sin^2 \theta)^4 - (1+4c\sin^2 \theta + 6c^2 \sin^4 \theta + 4c^3 \sin^6 \theta)] d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} \\
& = \frac{1}{c^4} \int_0^{\frac{1}{2}\pi} \frac{(1+\sin^2 \theta)^3 d\theta}{1'(1-e^2 \sin^2 \theta)} - \frac{1}{c^4} \int_0^{\frac{1}{2}\pi} \frac{(1+4c\sin^2 \theta + 6c^2 \sin^4 \theta + 4c^3 \sin^6 \theta) d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} \\
& = \frac{1}{15c^4 e^6} [(8c^3 + 3c^3 e^2 + 4c^3 e^4 - 10c^2 e^2 - 5c^2 e^4 + 15ce^4 - 15e^6) F(e, \tfrac{1}{2}\pi) \\
& \quad + 15e^6 II(e, c, \tfrac{1}{2}\pi) - (8c^3 + 7c^3 e^2 + 8c^3 e^4 - 10c^2 e^2 - 10c^2 e^4 + 15ce^4) E(e, \tfrac{1}{2}\pi)] \dots (79).
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\frac{1}{2}\pi} \frac{\sin^6 \theta \cos^2 \theta d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} = \int_0^{\frac{1}{2}\pi} \frac{(\sin^6 \theta - \sin^8 \theta) d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} \\
& = \frac{1}{15c^4 e^6} [(8c^3 - 3c^3 e^2 - 2c^3 e^4 - 10c^2 e^2 + 5c^2 e^4 + 15ce^4) E(e, \tfrac{1}{2}\pi) \\
& \quad - 15e^6(c+1) II(e, c, \tfrac{1}{2}\pi) - (8c^3 - 7c^3 e^2 - c^3 e^4 - 10c^2 e^2 \\
& \quad + 10c^2 e^4 + 15ce^4 - 15ce^6 - 15e^6) F(e, \tfrac{1}{2}\pi)] \dots \dots \dots (80).
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\frac{1}{2}\pi} \frac{\sin^4 \theta \cos^4 \theta d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} = \int_0^{\frac{1}{2}\pi} \frac{(\sin^4 \theta \cos^2 \theta - \sin^6 \theta \cos^2 \theta) d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} \\
& = \frac{1}{15c^4 e^6} [15e^6(c+1)^2 II(e, c, \tfrac{1}{2}\pi) - (8c^3 - 13c^3 e^2 + 3c^3 e^4 - 10c^2 e^2 + 20c^2 e^4 \\
& \quad + 15ce^4) E(e, \tfrac{1}{2}\pi) + (8c^3 - 17c^3 e^2 + 9c^3 e^4 - 10c^2 e^2 + 25c^2 e^4 \\
& \quad + 15ce^4 - 30ce^6 - 15c^2 e^6 - 15e^6) F(e, \tfrac{1}{2}\pi)] \dots \dots \dots (81).
\end{aligned}$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 \theta \cos^6 \theta d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)} = \int_0^{\frac{1}{2}\pi} \frac{(\sin^2 \theta \cos^4 \theta - \sin^4 \theta \cos^4 \theta) d\theta}{(1+\sin^2 \theta)_{1'}(1-e^2 \sin^2 \theta)}$$

$$\begin{aligned}
&= \frac{1}{15c^4e^6} [(8c^3 - 23c^3e^2 + 23c^3e^4 - 10c^2e^2 + 35c^2e^4 + 15ce^4)E(e, \tfrac{1}{2}\pi) \\
&\quad - 15e^6(c+1)^3II(e, c, \tfrac{1}{2}\pi) - (8c^3 - 27c^3e^2 + 34c^3e^4 - 10c^2e^2 \\
&\quad + 40c^2e^4 + 15ce^4 - 15c^3e^6 - 45c^2e^6 - 15e^6)F(e, \tfrac{1}{2}\pi)] \dots \dots \dots (82).
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{1}{2}\pi} \frac{\cos^8\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{(\cos^6\theta - \cos^6\theta\sin^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} \\
&= \frac{1}{15c^4e^6} [15e^6(c+1)^4II(e, c, \tfrac{1}{2}\pi) - (8c^3 - 33c^3e^2 + 58c^3e^4 - 10c^2e^2 + 50c^2e^4 \\
&\quad + 15ce^4)E(e, \tfrac{1}{2}\pi) + (8c^3 - 37c^3e^2 + 74c^3e^4 - 10c^2e^2 + 55c^2e^4 \\
&\quad + 15ce^4 - 60c^3e^6 - 90c^2e^6 - 60ce^6 - 15e^6)F(e, \tfrac{1}{2}\pi)] \dots \dots \dots (83).
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{1}{2}\pi} \frac{\sin^{10}\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{c^5} \int_0^{\frac{1}{2}\pi} \frac{(1+c\sin^2\theta)^4 d\theta}{\sqrt{(1-e^2\sin^2\theta)}} \\
&\quad - \frac{1}{c^5} \int_0^{\frac{1}{2}\pi} \frac{(1+5c\sin^2\theta + 10c^2\sin^4\theta + 10c^3\sin^6\theta + 5c^4\sin^8\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}} \\
&= \frac{1}{105c^5e^8} [(48c^4 + 16c^4e^2 + 17c^4e^4 + 24c^4e^6 - 56c^3e^2 - 21c^3e^4 - 28c^3e^6 \\
&\quad + 70c^2e^4 + 35c^2e^6 - 105ce^6 + 105e^8)F(e, \tfrac{1}{2}\pi) - (48c^4 + 40c^4e^2 \\
&\quad + 40c^4e^4 + 48c^4e^6 - 56c^3e^2 - 49c^3e^4 - 56c^3e^6 + 70c^2e^4 + 70c^2e^6 \\
&\quad - 105ce^6)E(e, \tfrac{1}{2}\pi) - 105e^8II(e, c, \tfrac{1}{2}\pi)] \dots \dots \dots (84).
\end{aligned}$$

As the foregoing are ample for illustration, we will proceed to other considerations.

[To be Continued.]